Sampling for (sparse) exchangeable random graphs

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Background

Problem of Denseness

- "Nice" statistical network models generate dense graphs
- Real-world networks are sparse

Projective + Exchangeable

- Challenge: seems that projective + exchangeable $\Rightarrow$ dense
- Projective: intuitively, collecting more data doesn't change the observation you already have
- Exchangeable: intuitively, vertex labels don't carry information about graph structure
- Lots of recent work using alternative notions of exchangeability, or abandoning it altogether

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Big Picture

- Finite vs infinite exchangeability is critical
- How are graphs of different size related?
- Understand this by *sampling schemes* associated with models

Graphon Generative Model

Graphon / Dense Exchangeable

Given a *graphon* $W : \mathbb{X}^2 \to [0,1]$, sample a random graph of n vertices:

1. Assign each vertex i a latent feature $x_i \in \mathbb{X}$ independently
2. Given the features, include each edge (i, j) independently with probability $W(x_i, x_j)$

Examples

- Stochastic block models (and descendants)
- Latent space models
- Many approaches to probabilistic matrix factorization, topic modeling, feature allocation

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Graphex Model (VR15, BCCH16)

Graphex / Sparse Exchangeable

Given a *graphex* $W : \mathscr{X}^2 \to [0,1]$, sample a size $s \in \mathbb{R}_+$ graph by:

1. Sample features: a Poisson process $\{(\theta_i, x_i)\}$ on $[0,s] \times \mathscr{X}$.
2. Given the features, include edge (θ_i, θ_j) with probability $W(x_i, x_j)$.
3. Include θ_i as a vertex whenever θ_i participates in at least one edge.

Examples

- Dense exchangeable / graphon models (essentially)
- Caron & Fox 2014 and related models
- Sparse graphs, power law degree distributions, small world behaviour

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Exchangeability

Dense Exchangeable Models

Dense Exchangeable

- Basic object: infinite random binary matrix (X_{ij})
- Random graphs: $(G_n)_{n\in\mathbb{N}}$ defined by taking adjacency matrix to be upper left $n \times n$ submatrix of (X_{ij})

Joint exchangeability of infinite random matrices

$(X_{ij}) \stackrel{d}{=} (X_{\sigma(i)\sigma(j)})$ for all permutations $\sigma \in S_\infty$ of the positive integers

Aldous–Hoover

A random adjacency matrix is exchangeable if and only if it's generated by some (dense) graphon

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Caron & Fox 2014: (Sparse) Exchangeable Graphs

Key insights

- adjacency matrix → adjacency measure
- matrix exchangeability → point process exchangeability

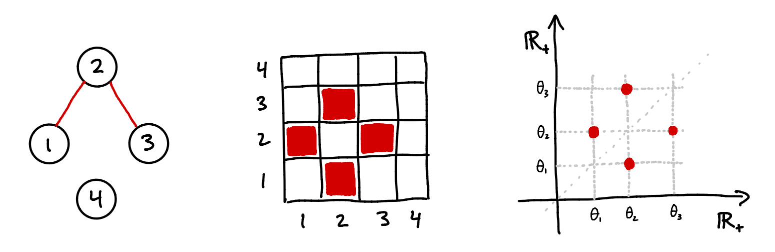

Figure: graph, adjacency matrix, and adjacency measure

Questions?

- Sparse and Dense exchangeable graphs are both *finitely* vertex exchangeable
- How do we escape Aldous–Hoover?
- What is the key difference between these models?

Sampling

Sampling

Dense Case

To generate a k vertex graph according to W:

1. Generate a n vertex graph generated by W
2. Sample k vertices at random and return the induced subgraph

p-sampling

Definition (p-sampling)

A p-sampling of a graph G is a random subgraph given by selecting each vertex of G independently with probability p, and then returning the induced edge set.

Theorem (VR16)

Let $(G_s)_{s \in \mathbb{R}_+}$ be generated by W. If $\mathrm{samp}(G_s, r/s)$ is an r/s-sampling of G_s, then

$$\mathrm{samp}(G_s, r/s) \overset{d}{=} G_r$$

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p-sampling defines graphex process models

Definition

Call $(G_s)_{s\in\mathbb{R}_+}$ an unlabeled random graph process indexed by $\mathbb{R}_+$ if, for all s, G_s is a finite unlabeled graph, and, for all $s \le t$, it holds that $G_s \subseteq G_t$ in the sense that there is some subgraph of G_t that is isomorphic to G_s.

Theorem (BCC**V**17)

Let $(G_s)_{s\in\mathbb{R}_+}$ be an unlabeled random graph process such that $e_s \uparrow \infty$ a.s. as $s \to \infty$. For each $s \in \mathbb{R}_+$ and $p \in (0,1)$, let $\mathrm{Smpl}_p(G_s)$ be a p-sampling of G_s. If for all $s \in \mathbb{R}_+$ and $p \in (0,1)$ it holds that

$$\mathrm{Smpl}_p(G_s) \overset{d}{=} G_{ps},$$

then there is some (possibly random, possibly non-integrable) almost surely non-zero graphex $\mathcal{W}$ that generates $(G_s)_{s\in\mathbb{R}_+}$.

p-sampling defines graphex process models

Exchangeability and Sampling

Sparse exchangeability is equivalent to *p*-sampling invariance

Exchangeable Models

- Dense: allows isolated vertices
- Sparse: doesn't.

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Summary

- Sparse exchangeability is equivalent to *p*-sampling invariance
- Key property of sparse exchangeable models: no isolated vertices
- Bonus result: exchangeability gives a general non-parametric consistent estimator for the graphex